Problems on Numerical Methods for Engineers

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CHAPTER 1

Arithmetic and Errors

Problem 1. The distance from the Earth to the Moon varies between $356400km$ and $406700km$. Give a bound on the absolute and relative errors incurred when using one of both values as the "real distance." This requires giving *two* bounds

Problem 2 (The Gregorian Calendar)**.** Explain the algorithm of the Gregorian Calendar, taking into account that

- A "real" year lasts 365.242374 days.
- It is designed so that the mismatch between the Spring Equinox and the 21st of March be never more than *dos días*.

Problem 3. The following integral

$$\int_0^1 e^{-x^2}\,dx$$

is computed using Taylor's polynomial of order 4 of the integrand, which is

$$T(e^{-x^2}, x = 0, 4) = 1 - x^2 + \frac{x^4}{2}.$$

Give an approximation to the absolute and relative errors incurred if the real value of the integral is 0.74682413+. ¿Are they remarkable?

Problem 4. When computing $1 - \sin(\pi/2 + x)$ for small x, one commits a specific type of error. Which one? Can this be prevented?

Problem 5. Compute the addition

$$\sum_{i=1}^{4} \frac{1}{7^i} = \frac{1}{7} + \frac{1}{7^2} + \frac{1}{7^3} + \frac{1}{7^4}$$

in the following ways:

- Truncating all the operations to 3 decimal digits.
- Rounding all the operations to 3 decimal digits.
- Truncating all the operations to 4 significant figures.
- Rounding all the operations to 4 significant figures.

Compute, in each case, the absolute and relative errors, if the exact value of the sum is 0.16659725.

Problem 6. In the second of the operations shown below, a number is truncated incurring (in this truncation) in an absolute error of 6.0×10^{-7} and a relative error of (approx.) 4.82×10^{-4}. Give the absolute and relative errors incurred when performing the second operation instead of the first.

$$26493 - \frac{33}{0.0012456}$$
$$26493 - \frac{33}{0.001245}$$

and explain.

Problem 7. Two values a and b are (approx.)

$$a = 1.742 \times 10^3, \quad b = 1.741 \times 10^3,$$

with absolute errors bound, respectively, by 1 and 0.5. Give bounds (if possible) for the absolute and relative errors in $a+b, ab, a/b$ and $a-b$ when using those values. If it is impossible to give a bound, explain why.

Problem 8. A laboratory watch makes an error (in binary digits) of 0.0000101 seconds every *pulse*. The watch pulses once every 3 seconds. How long will it take for it to be $1s$ out of sync? What is the relative error it incurs?

Problem 9. A register A stores values in 64−bit floating point and B stores 16−bit integer values without sign. During some process, one is only interested in the integer part of A. Is it reasonable to "just pass" this number into B? What is the maximum value B can store?

Problem 10. Assume the Euro-peseta exchange value is $1Eu = 166.386pts$. However, Law requires that these transactions be *rounded* to the nearest monetary unit (i.e. either 1 peseta or 1 cent). Compute

- The absolute and relative errors incurred when exchanging 1 Euro for 166 pesetas.
- The absolute and relative errors incurred when exchanging 1 peseta for its "equivalent" in Euros.
- The absolute and relative errors incurred when exchanging 1 cent for its "equivalent" in pesetas.

CHAPTER 2

Solutions to Nonlinear Equations

Problem 11. Use the Babilonian algorithm for square roots to compute $\sqrt{600}$ and $\sqrt{1000}$ so that the error when squaring be less than 0.1.

Problem 12. Explain the relation between the Babilonian algorithm for square roots and Newton-Raphson's.

Problem 13. Use *three* steps of Newton-Raphson's method for computing an approximation of $\sqrt{5}$, with $x_0 = 1$. Can you guarantee —without computing more steps— that $|x_6 - \sqrt{5}| < 0.0001$ [explain]?

Problem 14. What happens if one runs Newton-Raphson's algorithm for the function $f(x) = x^3 - x$ and seed $x_0 = \sqrt{1/5}$? Explain.

Problem 15. Compare the convergence speed of the Bisection algorithm in and Newton-Raphson's with seed 0.9 for $f(x) = x^7 - 0.9$ in $[0, 1]$.

Problem 16. Explain what happens if Newton-Raphson's algorithm is used with seed $x_0 = 3$ for $f(x) = \operatorname{atan}(x) - .3$. Can anything be done about it?

Problem 17. Perform two iterations of the secant algorithm for $f(x) = \tan(x) + .5$ with seeds 0 and 0.1.

Problem 18. Compute with 5 exact digits (using Newton-Raphson's algorithm) a root of $\cos(3x) - x$. Does this question make sense if one cannot compute the values of trigonometric functions exactly? Can anything be done about it?

Problem 19. Compute approximate roots to $f(x) = \cos(3x) - x$ using Newton-Raphson. Make sure that at least 4 decimal digits are exact.

Problem 20. Consider the functions verifying

$$\frac{f(x)}{f'(x)} = 3x.$$

without solving the differential equation, explain what happens to Newton-Raphson's algorithm when applied to them.

Problem 21. Consider $g(x) = \pi + \frac{1}{2}\sin(\frac{x}{2})$ and $f(x) = g(x) - x$.

- Verify that f has a single root c and that it is between 0 and 2π.
- Compute an approximate value of c using Newton-Raphson's algorithm.
- Compute an approximate value of c with at least 5 exact figures using any algorithm and explaining why those are exact figures.

Problem 22. Try to use the fixed point algorithm to compute a root of $f(x) = \cos(x) - x$ betwen $x = 0$ and $x = 1$ (notice that the interval $[0, 1]$ will not be the most useful one). Compare its convergence speed with that of the bisection algorithm and Newton-Raphson's.

Problem 23. Let us describe the *regula falsi* algorithm as that in which, in order to approximate a root of a continuous function f in $[a, b]$ in which $f(a)f(b) < 0$, one starts with a and b and at each step, one substitutes either (depending on the sign of f) by the meeting point of OX with the line joining $(a, f(a))$ and $(b, f(b))$. Describe it with precision.

Problem 24. Could it happen that, at some step in Newton-Raphson's algorithm one had $|x_{n+1} - x_n| < 10^{-7}$ and also $f(x_{n+1}) > 1$? Why?

Problem 25. Let $f : [0, 2] \to [0, 2]$ be a differentiable function such that $f(0) = 1.5$, $f(1) = 0$ and $f(2) = 0.5$. Does it satisfy the fixed point convergence conditions? Has it got any fixed point? Why?

Problem 26. Explain why Newton-Raphson's algorithm applied to $f(x) = \text{atan}(x)$ *will always diverge* if $|x_0| > 10$. *Clue:* $\text{atan}(10) > \sqrt{2}$.

Problem 27. Use *two* steps of Newton-Raphson's method for computing an approximation of $\sqrt[3]{2}$, with $x_0 = 1$. Can you guarantee —without computing more steps— that $|x_3 - \sqrt[3]{2}| < 0.001$?

CHAPTER 3

Solutions to Linear Systems

Problem 28. Consider the following $n \times n$ square matrix:

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & \dots & 0 \\ 1 & 0 & -1 & 0 & \dots & 0 \\ 1 & 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & \dots & -1 \\ 1 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Perform the following:

- Compute its determinant.
- Explain to what kind of intermediate matrices Gauss' algorithm will give rise.
- Compute the LU decomposition of A.
- Solve the system $Ax = b$ in a more efficient way, for any b.

Problem 29. Explain what happens if Gauss' method with partial pivoting is applied to an $n \times n$ singular matrix. And what if no pivoting is used?

Problem 30. Consider a system of linear equations $Ax = b$ whose entries have been obtained from measurements with 4 digits of precision. The matrix A is such that $\kappa_\infty(A) = 400$. Comment.

Problem 31. Verify that the condition number for the infinity norm of the following $n \times n$ matrix

$$B_n = \begin{pmatrix} 1 & -1 & -1 & \dots & -1 \\ 0 & 1 & -1 & \dots & -1 \\ 0 & 0 & 1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

is $\kappa_\infty(B_n) = n2^{n-1}$. Is it well-conditioned? However, is it difficult to solve a system $B_n x = b$?

Problem 32. Is the determinant of a matrix related in any way to its condition number? Why?

Problem 33. Verify that the product of matrices can be performed *by blocks*. That is, if

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \; B = \begin{pmatrix} B_{11} & B_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

where A_{ij} and B_{ij} are matrix of adequate dimensions, then

$$A \times B = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}$$

Problem 34. Verify that if P_{ij} is the permutation matrix (i, j) of size $n \times n$ and A is an $n \times m$, then PA is A with *rows* i and j swapped. If B is an $m \times n$ matrix, then BP is B with *columns* i and j swapped.

Problem 35. Given a "strange triangular" matrix as follows,

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n-1} & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n-1} & 0 \\ a_{31} & a_{32} & \dots & 0 & 0 \\ a_{n1} & 0 & \dots & 0 & 0 \end{pmatrix}.$$

How would a system which had it as coefficient matrix be solved? Why?

Problem 36. Let us say that a *tetratriangular* matrix is a $2n \times 2n$ matrix which can be subdivided in upper triangular $n \times n$ matrices as follows:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

(where A_{ij} are the upper triangular matrices). How would a linear system whose coefficient matrix were tetratriangular be solved? Would it be difficult?

Problem 37. Compute the condition number for the inifinity norm of the following matrix:

$$\begin{pmatrix} 1 & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} & \\ 0 & 1 & 0 & \dots & 0 & 0 & \\ 0 & 0 & 1 & \dots & 0 & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & \\ 0 & 0 & 0 & \dots & 0 & 1 & \end{pmatrix}$$

(one obviously has to compute its inverse).

Problem 38. Let I be a 800×600 pixels grayscale image (with values from 0 to 1). Consider the "blur" transformation which takes the *gray* value of each pixel and changes it for a linear combination of the values of the adyacent pixels and itself, according to the box:

a_{11}	a_{12}	a_{13}
a_{21}	a_{22}	a_{23}
a_{31}	a_{32}	a_{33}

where a_{22} (the very pixel's coefficient) is larger than the sum of absolute values of the others.

- Describe the transformation matrix if I is seen as a (huge) vector.
- In order to compute the inverse operation (to *focus*), should one use Gauss-Seidel's algorithm or Jacobi's? What would be better: using any of these or LU factorization? Why?
- What are the conditions for this transform to be symmetric? And positive-definite?

Problem 39 (Cholesky Factorization)**.** Let A be an $n \times n$ symmetric, positive definite matrix (so that it is nonsingular and Gauss' method without pivoting works). Compute its LU factorization. All the elements on the diagonal of U are positive. Proceed as follows:

(1) Start at column $i = 1$ of U. Let a_i bit the elemnt (i, i) and $l_i = \sqrt{a_i}$. Let $U_0 = U$ and $L_0 = L$.
(2) It is easy to check that $U_{i-1} = D(l_i)U_i$, where D is the identity matrix but at (i, i) where it is l_i; U_i is the upper triangular matrix equal to U_{i-1} except that row i has all its elements divided by l_i (check these statements).
(3) Let $L_i = L_{i-1}D(l_i)$. This is an operation by columns: the $i-$th column of L_i is the one of L_{i-1} times l_i (and the remaining ones are the same). Check this.
(4) Notice (and check) that the elements on column i of L_i are the same as on row i of U_i. This is easy to verify from the structure of L and U (and hence of L_i and U_i).
(5) Increase i by one until $i > n$ and repeat all the steps.

At the end of the process one has $A = \tilde{L}\tilde{U}$, where L and U are lower and upper triangular respectively but also the rows of L are equal to the columns of U. That is, $A = \tilde{L}\tilde{L}^T$. This is called *Cholesky's factorization.* This is not the actual algorithm used to compute this factorization, however.

Problem 40 (Cholesky's Factorization (2))**.** Cholesky's factorization of a symmetric positive definite matrix A can be computed using a modified LU factorization algorithm:

- Instead of using the pivot as divisor, one uses its square root at each time.
- One does not get the rows of A on U but the rows of A divided by the square root of the pivot.

- On the diagonal of L one gets also the square root of the pivots and on its columns, the multipliers.
- As A is symmetric and positive definite, one gets $L = U^T$ at the end.

This is the way to compute Cholesky's factorization: it takes advantage of the properties of A, which need to be known in advance.

Problem 41. Compute the LU factorization of the following matrix:

$$\begin{pmatrix} -1 & 2 & 0 & 0 & \dots & 0 & 0 \\ 2 & -5 & 2 & 0 & \dots & 0 & 0 \\ 0 & 2 & -5 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -5 & 2 \\ 0 & 0 & 0 & 0 & \dots & 2 & -5 \end{pmatrix}$$

Problem 42 (1 point)**.** Compute the LU factorization of the following matrix:

$$\begin{pmatrix} 1 & 2 & 0 & 0 & \dots & 0 & 0 \\ 1 & 3 & 2 & 0 & \dots & 0 & 0 \\ 1 & 3 & 3 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 3 & 3 & 3 & \dots & 3 & 2 \\ 1 & 3 & 3 & 3 & \dots & 3 & 3 \end{pmatrix}$$

Problem 43. An $n \times n$ matrix has all its entries less than 1 and the same happens for its inverse, but for one which is n. Can the condition number for the infinity norm be bounded?

Problem 44. The infinity norm of a matrix $M\ is 0.1$. An iterative problem $x = Mx + c$ needs to be solved. The seed $x_0 = (1, \dots, 1)$ is used and one obtains x_1 having all its elements positive and less than 1. How many iterations are needed in order to obtain a precision of 10^{-6} on each coordinate? What statement have you used to give your answer?

Problem 45. A matrix A is strictly diagonal dominant (by rows) and, actually $|a_{ii}| \geq 2 \sum_{j \neq i} |a_{ij}|$ for each row i.

- Explain if Jacobi's algorithm for $Ax = b$ will converge or not, for any b.
- Compute the number of iterations required for the error on each coordinate to be less than 10^{-6} if the algorithm starts with the vector $(1, 0, \dots, 0)$.

Problem 46. Explain if Jacobi's method converges for the system

$$\begin{pmatrix} 10 & 1 & 0 \\ -1 & 10 & 0 \\ 0 & 1 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Take the seed$x_0 = (1, 2, 3)$ and compute (explaining your result) the number of iterations needed to get an error less than 10^{-5} on each coordinate.

Problem 47. Study the convergence of Jacobi's algorithm for a system $Ax = b$ with

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 2 \end{pmatrix}.$$

The same question for Gauss-Seidel's.

CHAPTER 4

Interpolation

Problem 48. Given four points on the plain, a degree two spline is to be built so that the derivative at the intermediate points is continous. Compute the ecuations which describe this problem (do not solve them) and explain whether the system is consisten and whether it has a unique solution or not. If it is not, give some extra equations which make it so.

Problem 49. Is it true that the greater number of points the nearer the Lagrange interpolation polynomial is to the graph of the function to be interpolated? Explain.

Problem 50. Would you use the Lagrange interpolation polynomial to approximate the function $f(x) = e^x$ given 5 values? Why?

Problem 51. Given a cloud of 1000 points (coordinates (x, y)), one wishes to interpolate it using least squares for the functions $f(x) = 1$, $g(x) = x$ y $h(x) = x^2$. Can it be done? What size is the linear system to be solved?

Problem 52. A cloud of 300 points (coordinates (x, y)) is such that there exists a function $f(x) = ae^{bx}$ for which the total quadratic error is 0.1. Given the cloud $(x, \log(y))$, the least squares interpolation for the functions $1, x$; is $2+3x$. Can it be that the total quadratic error of e^2e^{3x} for the original cloud be greater than 0.1. Why?

Problem 53. Compute the Lagrange basis polynomials for the following points: $(0, 2), (1, 3), (2, 4), (3, 10), (4, 20)$ and use them to compute the Lagrange interpolation polynomial of degree 4 passing through them.

Problem 54. Compute the Lagrange interpolation polynomial for the points $(0, 0), (1, 2), (2, 0)$.

Problem 55 (1 point)**.** Compute the Lagrange interpolation polynomial for the points $(0, 0)$, $(1, 1)$, $(2, 3)$.

Problem 56 (1 point)**.** A table shows the values at different times of the speeds of a rocket. What kind of interpolation would you use to join them? Why?

Problem 57. Write the system of linear equations corresponding to the least squares interpolation problem for the cloud $(0,1),(1,2),(2,0),(3,4),(5,-1)$ and the functions $f(x)=x$, $g(x)=x^2$, $h(x)=x^4$. Do not solve it.

Problem 58 (1 point)**.** Given the following table:

x	y
0	1
1.5	2
2	1.5
4	1.75

Compute the value of the linear interpolation at $x=3$. **Explain**.

Problem 59. The table below shows the output of the following experiment: a car, in uniformly accelerated motion, starts moving at an initial speed v_0 and the space covered by it after different time intervals is recorded. Compute reasonable values for v_0 and a (the initial speed and the acceleration) which best explain the table. Units are implicit.

t	s
0.5	2.2
1.0	5.5
1.5	10.1
2.0	16.0

TABLA 1. Space versus time.

Problem 60. Coniser the cloud of points $(0,1)$, $(1,10)$, $(2,11)$, $(3,12),(12,13)$. Is it reasonable to assume that they follow a linear distribution? What kind of function would you use to interpolate by least squares? (There is not a unique answer). *Draw* the points before answering.

Problem 61. Consider the following data list:

x	y
-2	0.0338
-1.5	0.0397
-1	0.4119
-0.5	1.862
0	3.013
0.5	1.856
1	0.4240
1.5	0.0485
2	0.0249

And perform the following:

- Compute the table $(x, \log(y))$ (log is *always* the natural logarithm).
- Compute the linear interpolation of the new table using the functions 1 y x^2. Let $g(x)$ be the interpolation function.
- Let $f(x)$ be the function $e^{g(x)}$, that is, $e^a e^{bx^2}$, where a and b are the previous ones. Compute the total quadratic error of $f(x)$ for the *original* list.
- Compute the total quadratic error of $h(x) = 3e^{-2x^2}$ for the original list.
- Compare and comment the two last results.

Problem 62. A cloud of points represents the one-dimensional brownian motion of a particle. Would you use a cubic spline to plot the interpolating function? Why?

Problem 63. A cloud of points represents the instantaneous velocities of a car (on a *normal* drive) along a route. Would you use a cubic spline to plot the interpolating function? Why?

Problem 64. A data table shows the earnings of a gambler at the roulette along an evening. Would you interpolate them using a cubic spline or a linear interpolation? Why?

Problem 65. Does it make sense plotting the cubic spline of a data list of pairs $(age, income)$? Why? What would you use? Why?

Problem 66. State the equations required to solve the natural cubic spline which interpolates the following data table:

x	y
0	1
1	3
2	0
3	4
4	5

Same question for the *not-a-knot* spline. If you have a computer handy, solve the systems and plot both splines.

Problem 67 (1 point)**.** Given the following table:

x	y
0	1
1.5	2
2	1.5
4	1.75

Compute the value of the linear interpolation at $x = 3$. **Explain**.

Problem 68 (1 point)**.** A table shows the values at different times of the earnings (or losses) of a gambler at a roulette table during an evening. Would you use a linear interpolation or a cubic spline to plot the graph? Why?

Problem 69. A cloud of 200 points is to be approximated by a function of the form $f(x) = a + b\sin(x) + c\cos(x)$. Explain what size the linear system of equations obtained has. Same question but $f(x)$ being a polynomial in x of degree 10.

Problem 70. A function $f : [0, 10] \to \mathbb{R}$ is four times differentiable and $|f^{4)}|(x) < 1$ for $x \in [0, 10]$. It is approximated by a natural cubic spline $s(x)$ with nodes at $x_i = i$ for $i = 0, \dots, 10$. What is the maximum error between $f(x)$ and $s(x)$. Give a bound for the difference between $\int_0^{10} f(x)\,dx$ and $\int_0^{10} s(x)\,dx$.

Problem 71. How many nodes are necessary between 0 and 2π so that the clamped cubic spline approximates the function $\sin(x)$ with an error of at most 10^{-6}?

Problem 72. Would you use a clamped or a natural cubic spline to approximate the function $\sin(x)$ between 0 and 2π? Why?

CHAPTER 5

Numerical differentiation and integration

Problem 73 (1 point)**.** Given the following table, give reasonable (which means, **explain why** they are reasonable) approximate values for the derivative of $f(x)$ at $x = 0.5$ and at $x = 0.6$.

x	$f(x)$
0.4	3
0.5	3.1
0.6	3.15
0.75	3.2

Problem 74. Given the following data list, which comes from a function $f(x)$:

1.03	1.04	1.05	1.07	1.09	1.13
2.5	2.8	2.9	3.1	2.95	2.97

Do the following:

- Give an approximate value for $f'(1.04)$.
- Give an approximate value for $f'(1.09)$.
- Give approximate values for $f'(1.03)$ and $f'(1.13)$.

Problem 75. Given $f(x) = \cos(x)$, do the following:

(1) Approximate $\int_0^1 f(x)\,dx$ using the midpoint, trapezoidal and Simpson's rules.
(2) Compute the "exact" value of the integral.
(3) Compute the absolute and relative errors incurred by each method. Which of the two is better in this case, the midpoint or the trapezoidal rule?

Problem 76. Consider $f(x) = \tan(x)$ between $x = -1$ and $x = 1.5$. Do the following:

(1) Approximate $\int_{-1}^{1.5} f(x)\,dx$ using the midpoint, trapezoidal and Simpson's rules. punto medio, trapecio y Simpson.
(2) Compute the "exact" value of the integral.
(3) Compute the absolute and relative errors incurred by each method. Which of the two is better in this case, the midpoint or the trapezoidal rule?

Problem 77. The density of a metallic bar is described by the function $f(x) = \sin(x)/x$, for x from 1 to 3. Do the following:

(1) Approximate the mass using the midpoint, trapezoidal and Simpson's rules.
(2) Same computation but dividing the interval into 3 subintervals and applying the composite rules in each case.

Problem 78. Let $f(x) = x^7 - 15x^3 + 10x$ for from -2 to 2. Compute approximations to the integral between those two endpoints using:

- The simple midpoint, trapezoidal and Simpson's rules.
- The composite rules dividing the interval first into 2 and then into 3 subintervals (that is, this part has to be done twice).
- Compute all the absolute and relative errors incurred in each case.

Problem 79. Compute, using the midpoint rule and Simpson's rule, approximations to the following integral

$$\int_0^1 e^x\, dx.$$

Compute the exact value of the integral and give the absolute and relative errors incurred with each method.

Problem 80 (1 point)**.** Compute the following integral using the simple midpoint, trapeze and Simpson's rules:

$$\int_0^{2\pi} \sin^2(x)\, dx.$$

The exact value is π. **Explain** what you would do to get a better approximation. (There are **no points** for the computation of the integral, this question deals only with the explanation).

Problem 81. Consider $f(x) = (1-x)x^2(x+1) = -x^4 + x^2$. Compute

$$\int_{-1}^{1} f(x)\, dx$$

by hand and using Simpson's, the Midpoint and the Trapezoidal rules (1 point). Explain what is happenning and what you can do to improve the result (1 point).

Problem 82. Consider $f(x) = 10(1-x)x^2(x+1) = -10x^4 + 10x^2$. Compute

$$\int_{-1}^{1} f(x)\, dx$$

by hand and using Simpson's and the Trapezoidal rules (0.75 point). Explain what is happenning and what you can do to improve the approximations (0.75 point).

CHAPTER 6

Differential Equations

Problem 83. Consider the following initial-value problems. For each of them, verify that the proposed solution is valid for any value of the constant and compute the constant suitable for the initial condition.

- $y' = 0.03y$ for $t \in [0, 1]$, with $y(0) = 1000$. Solution: $y(t) = ke^{0.03t}$.
- $y' = t/y$ for $t \in [0, 1]$, with $y(0) = 1$. Solution: $y(t) = \sqrt{t^2 + k}$.
- $y' = e^{-y}$ for $t \in [1, 3]$, with $y(1) = 0$. Solution: $y(t) = \ln(t+k)$.
- $y' = y/t^2$ for $t \in [1, 2]$, with $y(1) = 1$. Solution: $y(t) = ke^{-1/t}$.
- $y' = 2t$ for $t \in [0, 2]$, with $y(0) = 1$. Solution: $y(t) = t^2 + k$.
- $y' = y^2$ for $t \in [0, 1/2]$, with $y(0) = 1$. Solution: $y(t) = 1/(k - t)$.

Problem 84. For each of the problems in 83, do the following:

(1) Solve it (approximately) using Euler's algorithm with two steps of equal length.
(2) Compute the absolute and relative errors incurred at the last point.
(3) Solve with four steps of equal length and compare.

Problem 85. For each of the problems in exercise 83, do:

(1) Solve it using Euler's Modified Algorithm using two evenly spaced steps.
(2) Compute the absolute and relative errors at the last step.
(3) Same for four steps.

Problem 86. For each of the problems in exercise 83, do:

(1) Solve it using Heun's Algorithm using two evenly spaced steps.
(2) Compute the absolute and relative errors at the last step.
(3) Same for four steps.

Problem 87. Consider the differential equation:

$$y' = x^2 + y^2.$$

Can you decide whether the solutions are increasing/decreasing? Let $y(x)$ be the solution satisfying $y(0) = 1$. Is it concave or convex? Has it got any critical point?

Same questions for the solution with $y(0) = -1$.

Problem 88. Consider the differential equation:

$$y' = y.$$

and let $y(x)$ be the solution to the initial value problem $y(-1) = 2$. Is $y(x)$ increasing? Has it got any critical point? Is it concave or convex?

Same questions for the solution with $y(-1) = -2$.

What about the solution with $y(-1) = 0$?

Problem 89. Consider the differential equation:

$$y' = \sin(y).$$

Can you say anything about the points where its solutions have a maximum or a minimum? And about their concavity/convexity?

Problem 90. Consider the differential equation:

$$y' = (x-1)y.$$

And let $y_1(x)$ be the solution to the initial value problem $y(0) = -1$. Has it got any critical points? In case it has, are they local maxima, minima or inflection points?

Same questions for the solution such that $y(1) = 2$.

Problem 91. Compute one step of the approximate solution to the following initial value problem, using Heun's method with $h = 0.1$:

$$\begin{cases} \dot{x} = y & x(0) = 0 \\ \dot{y} = -x & y(0) = 1 \\ \dot{z} = 1 & z(0) = 0 \end{cases}$$

The true solution is $x(t) = \sin(t), y(t) = \cos(t), z(t) = t$. Explain what kind of trajectory this equation describes.

Problem 92. A cup full of tea is left cooling at the window, which is at 0°. It is known that it cools at a speed which —in adequate units— is half the temperature. It starts at 70°. Compute, using Euler's method with two steps, the approximate temperature after 2 seconds. If the right answer is 25.752°, what are the absolute and relative errors?

Problem 93. There is a mass of a specific material which is left to its own. It is known that it disintegrates at a rate which is 0.1 the amount of mass at each moment. The experiment starts with 1kg. Compute, using Euler's method with two steps, the amount of mass after 2 seconds. If the exact value is 0.819kg, what are the absolute and relative errors incurred?

Problem 94. A projectile is thrown in the air with an initial speed (in (x, y) coordinates) of $(1, 1)$ —in adequate units. The only active forces are gravity and friction. Use $g = 10$ and consider that friction is proportional to the speed but in the opposite direction, with proportionality constant 0.05. Use Heun's method (one step) to give an approximation to the *position* of the projectile after one second. Assume that the initial position is $(0, 0)$ in an adequate reference system.

Problem 95. Some salt is diluted in a pool of water of volume 1000l, at a concentration of 0.1g/l. Water flows in and out of the pool at 3l/s but the inflow contains salt at 0.3g/l. Compute the concentration of salt in the pool after 2s and how long will it take for it to be 0.2g/l. Perform also the first computation in an approximate way using Euler's and Heun's method with 2 steps. For each calculation, give the absolute and relative errors.

Problem 96. In a specific plot of land there live two species of insects, one which serves as food for the other, let us say ants and spiders. Ants reproduce exponentially at a rate 0.1 and only die as a consequence of being eaten, with a likelyhood of 0.07. Spiders die with a likelyhood of 0.05 and breed proportionally to the likelyhood of feeding, with a proportionality constant 0.03. Spiders are counted in units while ants are in thousands. Write down the appropriate ordinary differential equation describing this simplistic model and, using Euler's method, compute its state after 2 seconds using 2 steps, assuming that the initial populations are 30000 ants and 10 spiders. Perform a computer simulation with smaller steps and a longer time range.

Problem 97. The spread of an infectious disease can be simulated with the following (very simplistic) model (called the *SIR* model). Let $S(t)$ be the "susceptible population", that is, those individuals who can be infected, $I(t)$ the "infected population", those that have been infected but are not cured and $R(t)$ the "removed" population, those that were infected and are now cured (and will not be infected again); if the disease is mortal, $R(t)$ includes the deceased individuals. The flow of individuals, when happens, is $S \to I \to R$, obviously. Let us assume that *the population is constant*, that is $N = S(t) + I(t) + R(t)$ is a fixed number. Then the model can be stated as

$$\begin{aligned}\frac{dS}{dt} &= -\beta\frac{SI}{N}\\ \frac{dI}{dt} &= \beta\frac{SI}{N} - \gamma I\\ \frac{dR}{dt} &= \gamma I\end{aligned}$$

Using Euler's method, compute two steps for an initial state of $S = 10000$, $I = 20$, $R = 0$ and values $\beta = 0.02$, $\gamma = 0.05$, time intervals of 1. Perform a computer simulation for a long timespan of that same problem.